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Journal of Geometry and Physics 57 (2007) 1541–1547
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

On the conformal geometry of transverse Riemann–Lorentz manifolds

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Received 29 September 2006; received in revised form 22 December 2006; accepted 6 January 2007
Available online 12 January 2007

Abstract

Physical reasons suggested in [J.B. Hartle, S.W. Hawking, Wave function of the universe, Phys. Rev. D41 (1990) 1815–1834] for the *Quantum Gravity Problem* lead us to study *type-changing metrics* on a manifold. The most interesting cases are *Transverse Riemann–Lorentz Manifolds*. Here we study the conformal geometry of such manifolds.

MSC: 53C50; 53B30; 53C15

1. Preliminaries

Let M be a connected manifold, $\dim M = m \geq 2$, and let g be a symmetrical covariant tensor field of order 2 on M. Assume that the set Σ of points where g degenerates is not empty. Consider $p \in \Sigma$ and $(\mathbb{U}, x)$ a coordinate system around p. We say that g is a *transverse type-changing metric* on p if $d_p(\det(g_{ab})) \neq 0$ (this condition does not depend on the choice of the coordinates). We call (M, g) *transverse type-changing pseudoriemannian manifold* if g is transverse type-changing on every point of Σ. In this case, Σ is a hypersurface of M. Moreover, at every point p of Σ the radical subspace $\mathrm{Rad}_p(M)$ of T_pM (that is, the subspace of T_pM which is *g-ortogonal to the whole T_pM*) is one-dimensional, and it can be transverse or tangent to the hypersurface Σ. The *index* of g is constant on every connected component of $\mathbb{M} = M - \Sigma$, thus $\mathbb{M}$ is a union of connected pseudoriemannian manifolds. Locally, Σ separates two pseudoriemannian manifolds whose indices differ in one unit (so we call Σ *transverse type-changing hypersurface*, in particular Σ is orientable). The most interesting cases, at least from the physical point of view [2], are those in which Σ separates a riemannian part from a lorentzian one. We call these cases *transverse Riemann–Lorentz manifolds*.

Let $\tau \in C^\infty(M)$ be such that $\tau|_\Sigma = 0$ and $d\tau|_\Sigma \neq 0$. We say that (locally, around Σ) $\tau = 0$ *is an equation for* Σ. Given $f \in C^\infty(M)$, it holds: $f|_\Sigma = 0 \Leftrightarrow f = k\tau$, for some $k \in C^\infty(M)$. In what follows we shall use this fact extensively.

On $\mathbb{M}$ we have naturally defined all the objects associated to pseudoriemannian geometry, derived from the Levi-Civita connection. In [4–7,1], the extendibility of geodesics, parallel transport and curvatures have been studied. Our

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0393-0440/$ - see front matter © 2007 Published by Elsevier B.V.
doi:10.1016/j.geomphys.2007.01.003

aim in the present paper is to study the conformal geometry of transverse Riemann–Lorentz manifolds, including criteria for the extendibility of the *Weyl conformal curvature*.

Let (M, g) be a transverse Riemann–Lorentz manifold. First of all, note that we do not have any Levi-Civita connection ∇ defined on the whole M. However, we have [4] a unique torsion-free metric *dual connection*

$$\Box : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}^*(M)$$

on M defined by a *Koszul-like formula*. On $\mathbb{M}$ it holds $\Box_X Y(Z) = g(\nabla_X Y, Z)$, and thus the concepts derived from Levi-Civita connection ∇ (on $\mathbb{M}$) coincide with those derived from the dual connection $\Box$.

We say that a vectorfield $R \in \mathfrak{X}(M)$ *is radical* if $R_p \in \mathrm{Rad}_p(M) - \{0\}$ for all $p \in \Sigma$. Given a radical vectorfield $R \in \mathfrak{X}(M)$, $\Box_X Y(R)|_\Sigma$ only depends on $X|_\Sigma$ and $Y|_\Sigma$, thus we obtain the following well-defined map

$$II^R : \mathfrak{X}_\Sigma \times \mathfrak{X}_\Sigma \to C^\infty(\Sigma), (X, Y) \mapsto \Box_X Y(R).$$

Note that the II^R-orthogonal complement to $\mathrm{Rad}_p(M)$ is $T_p\Sigma$ ([7], 1(a)), thus $X \in \mathfrak{X}_\Sigma$ is tangent to Σ if and only if $II^R(X, R) = 0$.

Because of the properties of $\Box$, the restriction of II^R to vectorfields in $\mathfrak{X}(\Sigma)$ is a well-defined $(0, 2)$ symmetric tensor field $II^R_\Sigma \in S^2(\Sigma)$. Furthermore, since $\Box_X Y$ is a one-form on M and the radical is one-dimensional, the condition $II^R_\Sigma = 0$ does not depend on the radical vectorfield R. *A transverse Riemann–Lorentz manifold is said to be II-flat if* $II^R_\Sigma = 0$, *for some (and thus, for any) radical vectorfield* R. It turns out ([7] for transverse, [1] for tangent radical) that M is *II*-flat if and only if all covariant derivatives $\nabla_X Y$, for $X, Y \in \mathfrak{X}(M)$ tangent to Σ, smoothly extend to M. Moreover, in that case, $\nabla_X Y|_\Sigma$ only depends on $X|_\Sigma$ and $Y|_\Sigma$, thus we obtain another well-defined map

$$III^R : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \to C^\infty(\Sigma), (X, Y) \mapsto II^R(\nabla_X Y, R)$$

which is a $(0, 2)$ symmetrical tensorfield on Σ. *A transverse Riemann–Lorentz II-flat metric is said to be III-flat if* $III^R = 0$.

If the radical is tangent, $\nabla_R R$ becomes transverse [1]; therefore, in order that a *II*-flat metric becomes *III*-flat, the radical must be transverse. And we have the following result [7], concerning the extendibility of curvature tensors:

Theorem 1. *The covariant curvature K smoothly extends to M if and only if the radical is transverse and g is II-flat, while the Ricci tensor Ric smoothly extends to M if and only if the radical is transverse and g is III-flat.*

2. A Gauss formula for transverse Riemann–Lorentz manifolds

Let (M, g) be a transverse Riemann–Lorentz manifold with transverse radical.

Lemma 2. *There exists a unique (canonically defined) radical vectorfield R such that $II^R(R, R) = 1$.*

Proof. Given a radical vectorfield U, consider $R = \left(II^U(U, U)\right)^{-\frac{1}{3}} U$, which is a well-defined radical vectorfield (since the radical is transverse). Thus $II^R(R, R) = 1$. Furthermore, if $Z = fR$ is another radical vectorfield such that $II^Z(Z, Z) = 1$, then $1 = II^Z(Z, Z) = f^3 II^R(R, R) = f^3$, and consequently $f = 1$. ♣

Suppose that (M, g) is *II*-flat. As we said before, given $X, Y \in \mathfrak{X}(\Sigma)$, $\nabla_X Y$ is well-defined. Moreover, $\tan(\nabla_X Y) := \nabla_X Y - III^R(X, Y) R$ is indeed tangent to Σ, since

$$II^R(R, \tan(\nabla_X Y)) = III^R(X, Y) - III^R(X, Y) II^R(R, R) = 0.$$

Lemma 3. *If $X, Y \in \mathfrak{X}(\Sigma)$ and ∇^Σ is the Levi-Civita connection of (Σ, g_Σ), it holds:*

$$\nabla_X Y = \nabla^\Sigma_X Y + III^R(X, Y) R.$$

Proof. Let be $Z \in \mathfrak{X}(\Sigma)$. Since (M, g) is *II*-flat, $\nabla_X Y$ is well defined and it must hold $\Box_X Y(Z) = g(\nabla_X Y, Z) = g_\Sigma(\tan(\nabla_X Y), Z)$. On the other hand, $\Box$ has always a good restriction $\Box : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \to \mathfrak{X}^*(\Sigma)$, which must coincide with $\Box^\Sigma$, the unique torsion-free metric dual connection on (Σ, g_Σ). Since (Σ, g_Σ) is riemannian, it must hold $\Box^\Sigma_X Y(Z) = g_\Sigma\left(\nabla^\Sigma_X Y, Z\right)$, and the result follows. ♣

The existence of a canonical radical vectorfield leads to the following Gauss formula:

Proposition 4. *Let (M, g) be a transverse Riemann–Lorentz manifold with transverse radical and II-flat. Then Σ is "totally geodesic" in the sense that, if $X, Y, Z, T \in \mathfrak{X}(\Sigma)$ it holds:*

$$K(X, Y, Z, T) = K^{\Sigma}(X, Y, Z, T)$$

where K^{Σ} is the covariant curvature of Σ.

Proof. As we said in the proof of previous lemma we have, for $X, Y, Z, T \in \mathfrak{X}(\Sigma)$: $\Box_X Y(Z) = \Box_X^{\Sigma} Y(Z)$, where $\Box^{\Sigma}$ is the dual connection of (Σ, g_{Σ}). Moreover, since $\Box_X R(T) = -\Box_X T(R) = -II^R(X, T) = 0$, again previous lemma leads to

$$\Box_X(\nabla_Y Z)(T) = \Box_X\left(\nabla_Y^{\Sigma} Z + III^R(Y, Z)\, R\right)(T) = \Box_X^{\Sigma}\left(\nabla_Y^{\Sigma} Z\right)(T)$$

what gives the result. ♣

Corollary 5. *Let (M, g) be a transverse Riemann–Lorentz manifold with transverse radical. If $(\mathbb{M}, g)$ is flat, then (M, g) is III-flat and Σ is flat.*

Proof. If $K = 0$ then $\mathrm{Ric} = 0$. In particular, Ric extends to M, thus by Theorem 1, (M, g) is *III*-flat. By Proposition 4, Σ is flat. ♣

We now restate Theorem 9 of [5] in the following terms (the flatness of Σ, being a consequence of the corollary, needs not be included as an extra hypothesis):

Theorem 6. *Let (M, g) be a transverse Riemann–Lorentz manifold. Then, M is locally flat around Σ if and only if, around every singular point $p \in \Sigma$, there exists a coordinate system $(\mathbb{U}, x)$ such that $g = \sum_{i=0}^{m-1}\left(\mathrm{d}x^i\right)^2 + \tau\,(\mathrm{d}x^m)^2$, where $\tau = 0$ is a local equation for Σ.*

3. Conformal geometry and the extendibility of Weyl curvature

Let us consider a transverse Riemann–Lorentz manifold (M, g) and the family $\mathcal{C} = \left\{\mathrm{e}^{2f} g : f \in C^{\infty}(M)\right\}$. Take $\overline{g} = \mathrm{e}^{2f} g \in \mathcal{C}$. Then $(M, \overline{g})$ is also a transverse Riemann–Lorentz manifold, and $\overline{\Sigma} = \Sigma$. Moreover, for each singular point $p \in \Sigma$ the radical subspaces are the same: $\overline{\mathrm{Rad}}_p(M) = \mathrm{Rad}_p(M)$. We say that $(M, \mathcal{C})$ *is a transverse Riemann–Lorentz conformal manifold if some (and thus any) $g \in \mathcal{C}$ is transverse Riemann–Lorentz*. Let $(M, \mathcal{C})$ be a transverse Riemann–Lorentz conformal manifold. We say that $g \in \mathcal{C}$ *is conformally II-flat if $II_{\Sigma}^R = h g_{\Sigma}$, for some radical vectorfield R and some $h \in C^{\infty}(\Sigma)$*. This definition does not depend on R and, even more, it is conformal: if $\overline{g} = \mathrm{e}^{2f} g \in \mathcal{C}$, then it holds

$$\overline{II}_{\Sigma}^R = \mathrm{e}^{2f}\left\{II_{\Sigma}^R - Rf|_{\Sigma} g_{\Sigma}\right\}. \tag{1}$$

Thus we say that $(M, \mathcal{C})$ is *conformally II-flat if some (and thus, any) metric $g \in \mathcal{C}$ is conformally II-flat*.

Proposition 7. *A transverse Riemann–Lorentz conformal manifold $(M, \mathcal{C})$ is conformally II-flat if and only if around every singular point $p \in \Sigma$ there exist an open neighbourhood $\mathbb{U}$ in M and a metric $g \in \mathcal{C}$ which is II-flat on $\mathbb{U}$, that is $II_{\Sigma\cap\mathbb{U}} = 0$.*

Proof. Let $(\mathbb{U}, E)$ be an adapted orthonormal frame near $p \in \Sigma$ (that is, E_m is radical and $(E_1, \ldots, E_{m-1})$ are orthonormal) and $g \in \mathcal{C}$. If $\mathcal{C}$ is conformally *II*-flat, then there exists $h \in C^{\infty}(\Sigma)$ such that $II_{\Sigma}^{E_m} = h g_{\Sigma}$. Take $\widehat{h} \in C^{\infty}(\mathbb{U})$ any local extension of h (shrinking $\mathbb{U}$ if necessary). There exists $f \in C^{\infty}(\mathbb{U})$ (shrinking again $\mathbb{U}$ if necessary) satisfying $E_m f = \widehat{h}$ (since it is locally a first order linear equation), what gives on $\mathbb{U}$: $II_{\Sigma}^{E_m} = (E_m f)|_{\Sigma} g_{\Sigma}$. Let $\widehat{f} \in C^{\infty}(M)$ be any extension of (possibly a restriction of) f. Applying (1) to g and $\overline{g} := \mathrm{e}^{2\widehat{f}} g \in \mathcal{C}$ we have $\overline{II}_{\Sigma}^{E_m} = 0$.

To show the converse we start considering $g \in \mathcal{C}$. Since conformally *II*-flatness is a local condition, it suffices to take an arbitrary $p \in \Sigma$ and $\overline{g} = \mathrm{e}^{2\widehat{f}} g \in \mathcal{C}$ such that $\overline{g}$ is *II*-flat around p. Then, formula (1) applied to g and $\overline{g}$ shows that $II_p^{\xi} = (\xi f)\, g_p$, where $\xi \in \mathrm{Rad}_p(M) - \{0\}$. ♣

In what follows, we study *conformally II-flat Riemann–Lorentz conformal* structures with transverse radical. Let g and $\overline{g} = e^{2f} g \in \mathcal{C}$ be two transverse Riemann–Lorentz metrics which are *II*-flat. Formula (1) shows that $(Rf)|_\Sigma = 0$. The expression of $\mathrm{grad}_g(f)$ in an adapted orthonormal frame such that $R = E_m$ is $\mathrm{grad}_g(f) = \sum_{i=1}^{m-1} (E_i f) E_i + \tau^{-1} (Rf) R$, thus $\mathrm{grad}_g(f)$ extends to the whole M. Now a simple computation gives:

$$\overline{III}^R = e^{2f}\{III^R - II^R(\mathrm{grad}_g(f), R) g_\Sigma\}. \tag{2}$$

We say that $g \in \mathcal{C}$ is *conformally III-flat if it is II-flat (in order that* III^R *exists) and it holds* $III^R = k g_\Sigma$, *for some radical vectorfield R and some* $k \in C^\infty(\Sigma)$. Since *II*-flatness is not conformal, the above definition, although independent of R, cannot be conformal. However, it is conformal in the subset of *II*-flat metrics.

Definition 8. We say that a transverse Riemann–Lorentz conformal manifold $(M, \mathcal{C})$ with transverse radical is conformally *III*-flat if it is conformally *II*-flat and every $g \in \mathcal{C}$ which is *II*-flat on some open $\mathbb{U}$ of M is also conformally *III*-flat on $\mathbb{U}$.

Note that there may exist no conformally *III*-flat metric on a conformally *III*-flat manifold, simply because there may exist no *II*-flat metric there. However, since a conformally *III*-flat space is conformally *II*-flat, we deduce from Proposition 7 that there always exist locally *II*-flat metrics. Let us show that in fact there also exist locally *III*-flat metrics:

Proposition 9. *A transverse Riemann–Lorentz conformal manifold* $(M, \mathcal{C})$ *with transverse radical is conformally III-flat if and only if around every singular point* $p \in \Sigma$ *there exist an open neighbourhood* $\mathbb{U}$ *in* M *and a metric* $g \in \mathcal{C}$ *which is III-flat on* $\mathbb{U}$, *that is* $III_{\Sigma\cap\mathbb{U}} = 0$.

Proof. Consider $p \in \Sigma$ and $(\mathbb{U}, E)$ a *completely adapted orthonormal frame* (i.e., E_m is radical and $(E_1, \ldots, E_{m-1})$ are orthonormal and tangent to Σ). If $(M, \mathcal{C})$ is conformally *III*-flat, there exist $g \in \mathcal{C}$ which is *II*-flat on $\mathbb{U}$ (without loss of generality) and $k \in C^\infty(\Sigma \cap \mathbb{U})$, such that $III^{E_m} = k g_\Sigma$. Since the radical is transverse, we have $II^{E_m}_{mm} \neq 0$, thus $k_1 := \frac{k}{II^{E_m}_{mm}}$ is C^∞ on $\Sigma \cap \mathbb{U}$. As in Proposition 7 we can obtain $f \in C^\infty(\mathbb{U})$ such that $E_m f = \tau \widehat{k_1}$, where $\tau = g(E_m, E_m)$ and $\widehat{k_1} \in C^\infty(\mathbb{U})$ is any local extension of k_1. Since $(E_m f)|_\Sigma = 0$, we get $\mathrm{grad}_g(f) \in \mathfrak{X}(\mathbb{U})$ and we have $II^{E_m}(\mathrm{grad}_g(f), E_m) = \left(\tau^{-1} E_m f\right)_\Sigma II^{E_m}_{mm} = k$. Now, take any extension $\widehat{f} \in C^\infty(M)$ of (possibly a restriction of) f. Since g is *II*-flat, we deduce from (1) that $\overline{g} = e^{2\widehat{f}} g \in \mathcal{C}$ is also *II*-flat on $\mathbb{U}$. We also deduce that $\overline{g}$ is *III*-flat on $\mathbb{U}$.

To prove the converse, first observe that the hypothesis implies in particular that $(M, \mathcal{C})$ is conformally *II*-flat. Consider $p \in \Sigma$ and $g \in \mathcal{C}$, *II*-flat on a neighbourhood of p. By hypothesis, there exists $\overline{g} = e^{2f} g \in \mathcal{C}$ which is *III*-flat around p. Thus we deduce from (2) that $III^R = II^R(\mathrm{grad}_g(f), R) g_\Sigma$, so g is conformally *III*-flat. ♣

In what follows we shall assume that $\dim M = m \geq 4$. We now study the extendibility of *the Weyl tensor*, naturally defined on $(\mathbb{M}, \mathcal{C}_\mathbb{M})$. It is well-known that this tensor plays a main role in deciding when $\mathbb{M}$ is (locally) conformally flat, according to *Weyl Theorem: a pseudoriemannian conformal manifold is (locally) conformally flat if and only if the Weyl tensor vanishes identically* (see for instance the preliminaries of [3]). At the end of the paper we discuss the problem of establishing a modified version of Weyl Theorem for transverse Riemann–Lorentz conformal manifolds.

The *Weyl tensor* W on $(\mathbb{M}, g_\mathbb{M})$ can be defined as:

$$W := K - h \bullet g \in \mathcal{I}^0_4(\mathbb{M}),$$

where $h = \frac{1}{m-2}\left\{\mathrm{Ric} - \frac{\mathrm{Sc}}{2(m-1)} g\right\}$ is the *Schouten tensor*, Ric is the Ricci tensor and Sc is the scalar curvature associated with $(\mathbb{M}, g_\mathbb{M})$, and where:

$$\bullet : S^2(\mathbb{M}) \times S^2(\mathbb{M}) \to \mathcal{I}^0_4(\mathbb{M})$$

is the so-called *Kulkarni-Nomizu product*, given by

$$\theta \bullet \omega(x, y, z, t) := \det\begin{pmatrix} \theta(x, z) & \omega(x, t) \\ \theta(y, z) & \omega(y, t) \end{pmatrix} + \det\begin{pmatrix} \omega(x, z) & \theta(x, t) \\ \omega(y, z) & \theta(y, t) \end{pmatrix}.$$

If we pick $\overline{g} = \mathrm{e}^{2f} g \in \mathcal{C}$, then the Weyl tensor associated to $\left(\mathbb{M}, \overline{g}_{\mathbb{M}}\right)$ satisfies $\overline{W} = \mathrm{e}^{2f} W$, thus *the Weyl conformal curvature* $\mathcal{W} := \uparrow_2^1 W \in \mathcal{I}_3^1(\mathbb{M})$ becomes a conformal invariant. Notice that the extendibility of W (which is equivalent to the extendibility of $\mathcal{W}$) is a conformal condition, therefore it should be stated in terms of the conformal structure. In fact, we prove that it is equivalent to conformal *III*-flatness.

Theorem 10. *Let $(M, \mathcal{C})$ be a transverse Riemann–Lorentz conformal manifold, with* $\dim M = m \geq 4$. *Then W (smoothly) extends to the whole M if and only if the radical is transverse and $\mathcal{C}$ is conformally III-flat.*

Proof. If $(M, \mathcal{C})$ has transverse radical and is conformally *III*-flat, there exist (Proposition 9) a M-open covering $\{\mathbb{U}_\alpha\}$ of Σ and a family of metrics $\{g_\alpha\}$ in $\mathcal{C}$ such that g_α is *III*-flat on $\mathbb{U}_\alpha$. By Theorem 1, the covariant curvature K_α, the Ricci tensor Ric_α and the scalar curvature Sc_α associated to g_α extend to $\Sigma \cap \mathbb{U}_\alpha$, therefore the Weyl tensor W_α also extends to $\Sigma \cap \mathbb{U}_\alpha$. Since this is a conformal condition, W_α extends to $\Sigma \cap \mathbb{U}_\beta$ for all β, and thus W_α extends to the whole M.

To show the converse we start picking an adapted orthonormal frame $(\mathbb{U}, E)$. Then, we can express the functions $W_{abcd} = W(E_a, E_b, E_c, E_d)$ as second order polynomials in $\tau^{-1} = (g(E_m, E_m))$. Let us call $(W_{abcd})_0, (W_{abcd})_1$ and $(W_{abcd})_2$ the differentiable coefficients of the terms of order 0, 1 and 2. Since $\tau = 0$ is a local equation for Σ, W extends to $\mathbb{U}$ if and only if the restricted functions $(W_{abcd})_2|_\Sigma$ and $(W_{abcd})_1 + \tau^{-1}(W_{abcd})_2|_\Sigma$ identically vanish.

Suppose the radical is tangent to Σ at a singular point $p \in \Sigma$. We can choose the frame such that $E_1(p), E_2(p) \in T_pM - T_p\Sigma$. But then, using that $II^{E_m}(E_m, E_m)(p) = 0$ (because the radical is tangent), we obtain $(W_{1323}(p))_2 = \frac{\varepsilon_3}{m-2} II_p^{E_m}(E_1, E_m)\, II_p^{E_m}(E_2, E_m)$. Since E_1 and E_2 are transverse to Σ at p, $(W_{1323}(p))_2 \neq 0$, hence W cannot be extended. Therefore the radical must be transverse to Σ.

Once we know that the radical must be always transverse to Σ (thus $II_{mm}^{E_m} \neq 0$), we can choose the orthonormal frame $(\mathbb{U}, E)$ completely adapted. Thus, picking i, j, k different from m, with i, j different from k, and using $II_{im}^{E_m} = 0$, we have: if $i \neq j$, then $0 = \left(W_{ikjk}\right)_2|_\Sigma = -\frac{\varepsilon_k}{m-2} II_{ij}^{E_m} II_{mm}^{E_m}$. Since $II_{mm}^{E_m} \neq 0$, we get $II_{ij}^{E_m} = 0$. If $i = j$ (and using $II_{ij}^{E_m} = 0$), the $\binom{m-1}{2}$ equalities $0 = (W_{ikik})_2|_\Sigma$, suitably manipulated, give us $\varepsilon_i II_{ii}^{E_m} + \varepsilon_k II_{kk}^{E_m} = \frac{2C}{m-1}$, where $C = \sum_{l=1}^{m-1} \varepsilon_l II_{ll}^{E_m} \in C^\infty(\mathbb{U})$. Subtracting the equation for i, k from the equation for k, j, we obtain $\varepsilon_i II_{ii}^{E_m} - \varepsilon_j II_{jj}^{E_m} = 0$, thus $\varepsilon_i II_{ii}^{E_m} = \varepsilon_j II_{jj}^{E_m}$. Defining $k := \varepsilon_1 II_{11}^{E_m} \in C^\infty(\Sigma \cap \mathbb{U})$, it holds $II_{ii}^{E_m} = \varepsilon_i \varepsilon_1 II_{11}^{E_m} = k g_{ii}$ and $II_{ij}^{E_m} = 0 = k g_{ij}$ (where $i \neq j$), what means $II_\Sigma^{E_m} = k g_\Sigma$, that is, g is conformally *II*-flat on $\mathbb{U}$, and therefore $(M, \mathcal{C})$ is conformally *II*-flat.

Once we know that $(M, \mathcal{C})$ is conformally *II*-flat, we can choose a metric $g \in \mathcal{C}$ which is *II*-flat on $\mathbb{U}$ (shrinking $\mathbb{U}$ if neccesary). By Theorem 1, the covariant curvature K associated with g extends to $\Sigma \cap \mathbb{U}$ and, since W also does it, necessarily $h \bullet g$ extends to $\Sigma \cap \mathbb{U}$. Picking i, j, k different from m, with i, j different from k, we get $(h \bullet g)_{ikjk} = \varepsilon_k h_{ij} + \delta_{ij} \varepsilon_i h_{kk} = A_{ijk} + \tau^{-1} B_{ijk}$, therefore the function

$$B_{ijk} := \frac{1}{m-2} \left\{ \varepsilon_k K_{imjm} + \delta_{ij} \varepsilon_i K_{kmkm} - \frac{2\varepsilon_k \delta_{ij} \varepsilon_i}{m-1} \sum_{l=1}^{m-1} \varepsilon_l K_{lmlm} \right\}$$

must vanish on Σ. Using the same argument as before, but with the equalities $0 = B_{ijk}|_\Sigma$, we get: $III^{E_m} = k g_\Sigma$, where $k := \varepsilon_1 III_{11}^{E_m} \in C^\infty(\Sigma \cap \mathbb{U})$, that is g is conformally *III*-flat on $\mathbb{U}$, and thus $(M, \mathcal{C})$ is conformally *III*-flat. ♣

Let us consider the following conjecture:

Conjecture 11. *Let $(M, \mathcal{C})$ be a transverse Riemann–Lorentz conformal manifold, with* $\dim M = m \geq 4$. *A necessary condition for being $W = 0$ is that, around every singular point $p \in \Sigma$, there exist a coordinate system $(\mathbb{U}, x)$ and a metric $g \in \mathcal{C}$ such that $g = \sum_{i=0}^{m-1} \left(\mathrm{d}x^i\right)^2 + \tau\left(\mathrm{d}x^m\right)^2$, where $\tau = 0$ is a local equation for Σ.*

Using Theorem 6, it becomes obvious that the necessary condition stated in the conjecture is always sufficient for having $W = 0$ around Σ.

If the conjecture is true, Σ must be (locally) conformally flat, which is well known equivalent to either $W^\Sigma = 0$ (if $m > 4$) or $\nabla_X^\Sigma h^\Sigma(Y, Z) = \nabla_Y^\Sigma h^\Sigma(X, Z)$ (if $m = 4$). But the extendibility of W, equivalent (Theorem 10) to conformal *III*-flatness, implies (Proposition 9) the existence of a metric $g \in \mathcal{C}$ which is *III*-flat around Σ, thus satisfying (Proposition 4):

$$W|_{T\Sigma} = (K - h \bullet g)|_{T\Sigma} = K^{\Sigma} - h|_{T\Sigma} \bullet g_{\Sigma} = W^{\Sigma} + (h^{\Sigma} - h|_{T\Sigma}) \bullet g_{\Sigma}.$$

Because conditions $W = 0$ and $W^{\Sigma} = 0$ are conformal, any counterexample $(M, \mathcal{C})$ to the above conjecture must admit a metric $g \in \mathcal{C}$ which is *III*-flat around Σ and satisfies either $h^{\Sigma} \neq h|_{T\Sigma}$ (if $m > 4$) or (Lemma 3) $\nabla_X h(Y, Z) \neq \nabla_Y h(X, Z)$, for some $X, Y, Z \in \mathfrak{X}(\Sigma)$ (if $m = 4$). Now a straightforward computation for *III*-flat metrics, using an orthonormal completely adapted frame, leads to the following expression in terms of extendible quantities:

$$h^{\Sigma}_{ij} - h_{ij}|_{T\Sigma} = \frac{-1}{m-2}\left\{\frac{K_{imjm}}{\tau} - \frac{1}{m-3}\sum_{l=1}^{m-1} K_{iljl} - \frac{1}{m-1}\left[\sum_{k=1}^{m-1}\frac{K_{kmkm}}{\tau} - \frac{1}{m-3}\sum_{k,l=1}^{m-1} K_{klkl}\right]\delta_{ij}\right\}\Bigg|_{\Sigma},$$

$(i, j = 1, \ldots, m-1)$, which shows that the construction of counterexamples is not easy.

In fact, *the conjecture is true for transverse Riemann–Lorentz warped products*, as we show right now. Let us consider a m-dimensional ($m \geq 4$) transverse Riemann–Lorentz manifold (M, g) of the form $M = I \times S$, where $\dim I = 1$, $0 \in I$, and $g = f(t)^2 g_S - t\mathrm{d}t^2$, where $f \in C^{\infty}(I)$, $f > 0$ and g_S is riemannian (we identify t, f and g_S with the corresponding pullbacks by the canonical projections). Thus $\Sigma = \{0\} \times S$ is homothetic to S with scale factor $f(0)$. Calling $U \in \mathfrak{X}(M)$ the (nowhere zero) lift of the vectorfield $\frac{\mathrm{d}}{\mathrm{d}t} \in \mathfrak{X}(I)$, one immediately sees that U is radical and transverse to Σ. It is not difficult to compute the curvature tensors on $\mathbb{M}$. Standard results on warped products (see [8], Chapter 7) lead to (we denote by $X, Y \in \mathfrak{X}(M)$ the lifts of corresponding vectorfields $\overline{X}, \overline{Y} \in \mathfrak{X}(S)$): $\nabla_U U = \frac{1}{2t} U$, $\nabla_U X = \nabla_X U = \frac{f'}{f} X$ and $\nabla_X Y = g(X, Y)\frac{f'}{tf} U + \nabla^S_{\overline{X}} \overline{Y}$ (where ∇^S is the Levi-Civita connection on S and $\nabla^S_{\overline{X}} \overline{Y}$ is the lift of the corresponding vectorfield on S) and also to the following expressions for the curvature tensors:

$$\begin{cases} K = f^2 K^S + \dfrac{f'^2 f^2}{2t} g_S \bullet g_S + \dfrac{f}{2}\left(\dfrac{f'}{t} - 2f''\right) g_S \bullet \mathrm{d}t^2 \\ \mathrm{Ric} = \mathrm{Ric}^S - \left(\dfrac{f}{2t}\left(\dfrac{f'}{t} - 2f''\right) - (m-2)\dfrac{f'^2}{t}\right) g_S + \dfrac{m-1}{2f}\left(\dfrac{f'}{t} - 2f''\right)\mathrm{d}t^2 \\ \mathrm{Sc} = \dfrac{\mathrm{Sc}^S}{f^2} - \dfrac{m-1}{f^2}\left(\dfrac{f}{t}\left(\dfrac{f'}{t} - 2f''\right) - (m-2)\dfrac{f'^2}{t}\right) \\ h = \dfrac{m-3}{m-2} h^S + \left(\dfrac{\mathrm{Sc}^S}{2(m-2)^2(m-1)} + \dfrac{f'^2}{2t}\right) g_S \\ \quad + \left(\dfrac{t\,\mathrm{Sc}^S}{2(m-1)(m-2)f^2} + \dfrac{1}{2f}\left(\dfrac{f'^2}{f} + \dfrac{f'}{t} - 2f''\right)\right)\mathrm{d}t^2 \\ W = f^2 W^S + \dfrac{1}{(m-2)}\left(\mathrm{Ric}^S - \dfrac{\mathrm{Sc}^S}{m-1} g_S\right) \bullet \left(\dfrac{f^2}{m-3} g_S + t\mathrm{d}t^2\right) \end{cases}$$

(K^S, Ric^S, Sc^S, h^S and W^S denote of course the pullbacks by the projection of the corresponding tensor fields on S). It follows:

Lemma 12. *The following three conditions are equivalent:* (1) K *extends to* M, (2) $f'(0) = 0$ *and* (3) h *extends to* M. *Also the following are equivalent:* (1) Ric *extends to* M, (2) $(f'/t)(0) = 0$ *and* (3) Sc *extends to* M. *Moreover,* W *extends to* M *in any case.*

The fact that W extends to M was obvious from the very beginning: the map $\Psi \equiv \psi \times \mathrm{id} : (I - \{0\}) \times S \to \mathbb{R} \times S$, given by $T \equiv \psi(t) := \int_0^t \frac{|s|^{\frac{1}{2}}\mathrm{d}s}{f(s)}$, is a conformal diffeomorphism onto its (non-connected) image with the metric $\overline{g} \equiv -(\mathrm{d}T)^2 + g_S$, thus it preserves the $\binom{1}{3}$-Weyl tensors, and since $\overline{g}$ is regular around $T = 0$ and $f(0) \neq 0$, $\overline{W}$ (and therefore W) extends to the whole M. It follows from Theorem 10 that the conformal manifold $(M, [g])$ is (in any case) conformally *III*-flat.

Lemma 13. *The following four conditions are equivalent:* (1) $W = 0$, (2) $W^S = 0 = \mathrm{Ric}^S - \frac{\mathrm{Sc}^S}{m-1} g_S$ *and* (3) Σ *has constant (sectional) curvature.*

Proof. (1) $\Leftrightarrow$ (2) follows from the above formula. (2) $\Rightarrow$ (3): $\mathrm{Ric}^S - \frac{\mathrm{Sc}^S}{m-1} g_S = 0$ implies (Schur's lemma) $\mathrm{Sc}^S = (m-1)(m-2)C$ (constant), thus $h^S = \frac{C}{2} g_S$; moreover $W^S = 0$ leads to $K^S = \frac{C}{2} g_S \bullet g_S$. (3) $\Rightarrow$ (2): From $K^S = \frac{C}{2} g_S \bullet g_S$, one immediately gets: $W^S = 0 = \mathrm{Ric}^S - \frac{\mathrm{Sc}^S}{m-1} g_S$. ♣

Proposition 14. *The Conjecture 11 is true for any transverse Riemann–Lorentz conformal manifold $(M, \mathcal{C})$ such that some $g \in \mathcal{C}$ is a warped product.*

Proof. Let $g = f(t)^2 g_S - t\mathrm{d}t^2 \in \mathcal{C}$ be a transverse warped product metric on $M = I \times S$. Note that $g = f(t)^2 \left\{ g_S - \frac{t}{f(t)^2} \mathrm{d}t^2 \right\}$. From $W = 0$ and Lemma 13 we get, around any $p \in \Sigma$, coordinates $(\mathbb{V}, y)$ of Σ such that $f(0)^2 g_S = g_\Sigma = \mathrm{e}^{2h} \sum_{i=1}^{m-1} \left(\mathrm{d}y^i\right)^2$, for some $h \in C^\infty(\Sigma)$. Choosing $x^i := y^i \circ \pi$, $x^m := t$ and $\tau := \frac{-t\mathrm{e}^{-2h}}{f(t)^2}$, we get $g = \mathrm{e}^{2h} f(t)^2 \left\{ \sum_{i=1}^{m-1} \left(\mathrm{d}x^i\right)^2 + \tau \left(\mathrm{d}x^m\right)^2 \right\}$, and we are finished. ♣

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